A Problem is Something You Don’t Want to Have\(^1\) … or Do You?

The Nature of Mathematical Tasks

Susan N. Friel with Tracy Goodson-Epsy, Jane Gleason and Tery Gunter

Teaching and learning mathematics is all about solving problems. Classroom instruction is generally organized and orchestrated around mathematical problems or tasks. As a teacher, more often than not, you choose the problems. You may rely on textbooks or other sources for problems or you may create your own problems for students to solve. Sometimes you have students pose their own problems for other classmates to solve. Whatever way tasks are selected and/or designed, as teachers, we want our students to engage with them in ways that are meaningful and that motivate learning mathematics. However, all too often, students may view the tasks we use in mathematics lessons more as exercises that they must work through, without much thought or involvement, rather than as intellectual challenges with which we would like to engage them. There probably is no decision that we make that has greater impact on our students’ opportunities to learn and on their perceptions of what mathematics is all about than the selection or creation of tasks we will use to engage our students in the study of mathematics (Lappan & Briars, 1995).

George Polya (1973/1945) recognized the importance of problems in mathematics. He notes that (p. v):

…a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to

\(^1\) Title taken from Outhred & Sardelich (2005).
solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.

In this day and age of accountability, time is certainly a factor. How we choose to fill the time our students use for their mathematics lessons has never been more important!

Polya’s interest was in helping students solve more complex (and interesting!) mathematics problems. He recognized that problems or tasks are what mathematics is about and that there are different kinds of problems that may be used to learn and teach mathematics. Not all tasks are created equal, and different tasks do provoke different levels and kinds of student thinking. Clearly, the level and kinds of thinking in which students engage determine what they will learn. So, how might we go further in exploring the nature of mathematical tasks and their use?

It is possible to make distinctions among the goals of mathematical tasks (Stein, Smith, Henningsen, & Silver, 2000) and the kinds of mathematical thinking that they require. For some mathematics tasks the focus is to perform a memorized procedure in a routine manner; these kinds of tasks lead to one type of opportunity for thinking. For example, what do you need to know and be able to do to solve the problems below (Figure 1)? With what kind(s) thinking are you engaged?

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**Figure 1**

<table>
<thead>
<tr>
<th>Manipulatives/Tools Available: Pictures of Base 10 blocks OR Place Value Arrow Cards</th>
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1. What is another name for 345?
   a. 300 + 40 + 5
   b. 3000 + 400 + 50
   c. 3000 + 400 + 5
   d. 300 + 400 + 5

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<tr>
<td>a.</td>
<td>4000 + 500 + 90 + 3</td>
<td>b.</td>
<td>2000 + 90</td>
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<td>c.</td>
<td>3000 + 200</td>
<td>d.</td>
<td>8000 + 5</td>
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<td>e.</td>
<td>1000 + 80 + 7</td>
<td>f.</td>
<td>5000 + 600 + 9</td>
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<td>g.</td>
<td>6 hundred 4 thousand</td>
<td>h.</td>
<td>8 tens 4 thousand</td>
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<td>i.</td>
<td>3 ones 7 thousand 2 hundred</td>
<td>j.</td>
<td>4 hundred 5 ones 1 thousand</td>
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<td>k.</td>
<td>fifty 7, thousand</td>
<td>l.</td>
<td>4 thousand 5</td>
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<td>m.</td>
<td>9, sixty, 4 thousand</td>
<td>n.</td>
<td>8 hundred 3 thousand 9</td>
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Other mathematics tasks engage the solver with concepts and with making connections and lead to different kinds of opportunities for thinking. Look at the second example (Figure 2); what do you need to know and be able to do to solve this problem? With what kind(s) of thinking are you engaged now?

**Figure 2**

**Manipulatives/Tools Available: Base 10 blocks**

On a field trip to the produce market, Mrs. Johnson’s class counted 134 different kinds of produce. When they returned to school, Mrs. Johnson asked them to use place value blocks to show 134 in as many ways as they could think of. Callie, Jen, and Tucker showed 134 in different ways.

**Callie’s Way**

134 = 100 + 30 + 4 or 1 hundred 3 tens 4 ones

**Jen’s Way**

134 = 130 + 4
100 = 10 tens, and 30 = 3 tens, so 134 = 13 tens and 4 ones
Tucker’s Way

\[
134 = 100 + 30 + 4
\]

\[
30 = 2 \text{ tens} 10 \text{ ones}, \text{ and } 4 = 4 \text{ ones}, \text{ so}
\]

\[
134 = 1 \text{ hundred} 2 \text{ tens and} 14 \text{ ones}
\]

Use place value blocks to show each number in two ways. Draw the blocks you use for each answer.

a. 251

b. 301

c. How could skip counting by 10s help you know that Jen’s way is a correct way to rename 134?

Look at the third example (Figure 3); what do you need to know and be able to do to solve this problem? With what kind(s) of thinking are you engaged now?

Figure 3

Manipulatives/Tools Available: Base 10 blocks or other materials the child has worked with

1. Can you show me what (point to card with 342 written on it) means using these materials (have base ten materials or materials the child has worked with available)? If successful, continue by asking:

2. What does this part (point to the 4 – but don’t say the digit name) have to do with the pieces you used to show 342?

3. What does this part (point to the 3 – but don’t say the digit name) have to do with the pieces you used to show 342?

4. Can you think of another way to show 342 using these materials? How did you think about this?

5. Another way? How did you think about this?

6. How many different ways can you display 342? How did you think about this?

7. Which display has the most groups? Why?

8. Which display has the fewest groups? Why?
Take a moment to compare these three sets of tasks. Make list of similarities or differences among tasks.

All three sets of tasks involve developing place value understanding. The first task (Figure 1) involves using Base Ten blocks or Place Value Arrow cards to assist children in representing numerals, given values represented in expanded form or using word names. The second task (Figure 2) requires students to confront what it means to group by tens and how this relates to place value notation. Students must explore the question of whether the same number can be represented in different ways. The second set of tasks provides examples of such representations. The third task (Figure 3) requires students to describe their understandings of the place value concepts involved in the questions. The problem requires the solver to describe the meaning for each of the digits in 342, but more importantly, pushes the solver to consider if the 342 can be represented in more than one way and if the different representations used would change the number that is illustrated. One of the main differences between Task Two and Task Three is that the third task retains an open-ended approach while Task Two leads the students to particular representations based on the examples provided. Each task addresses the same essential concepts but the three tasks are very different in terms of the kinds of opportunities for thinking in which they may engage the student.

The mathematical tasks with which students engage form the basis of their opportunities to learn what mathematics is and how one does it (Doyle, 1984; 1988). Over time, the cumulative effect of classroom-based tasks is students' implicit development of ideas about the nature of mathematics and of how they engage with mathematics—about whether
mathematics is something that they personally can make sense of, and how long and how hard they should have to work to do so. Students form an internal view of mathematics as a formal discipline, form a sense of the role of human activity in mathematics, and develop an expectation for their own role in mathematical activity.

Below and on the next page is another set of two tasks (Figure 4). Try completing each task as directed. What do you need to know and be able to do to solve each? What might be the purpose for using each task with upper elementary or sixth grade students? What methods do you think students might use to solve each task? What misconceptions might surface? What errors? Do you have any preferences about which task would be more challenging? Why or why not?

**Figure 4: Which task is more challenging?**

1. You know that $3 \times 5 \times 7 = 105$. Use this fact to find
   a. $9 \times 5 \times 7 = $
   b. $= 3 \times 5 \times 14$
   c. $3 \times 50 \times 7 = $
   d. $\div 3 = 5 \times 70$
   e. $15 \times = 5 \times 21$
   f. $3500 \div 5 = \times 14 \times 5$
   g. $6 \times 5 \times 7 = 18 \times \div 3$
2. Complete the following multiplication facts in one minute or less:

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<td>9x3</td>
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<td>2x7</td>
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<tr>
<td>3x9</td>
<td>8x7</td>
<td>9x4</td>
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Stein and her colleagues (Stein, Smith, Henningsen, & Silver, 2000) developed a taxonomy of mathematical tasks that is based on the kinds and levels of thinking required to solve them. Mathematical tasks are viewed as having high cognitive demand\(^2\) when they engage students in the processes of active inquiry and validation (doing mathematics) or encourage them to use procedures in ways that are meaningfully connected to concepts or understanding (procedures with connections). Tasks that encourage students to use procedures, formulas, or algorithms in ways that disassociate them from meaning (procedures without connections), or that consist entirely of memorization or the reproduction of previously memorized facts (memorization) are categorized as having low cognitive demand tasks. Both may be useful in mathematics instruction but determining their value involves developing a framework for thinking about the nature and use of mathematical tasks in general.

If we look back to the two problems that involve knowledge of multiplication facts, it may be apparent that these two tasks fit within two task classification categories. In the first task, Stein, et. al. (2000) refer to tasks as high level tasks, meaning tasks with high cognitive demand, or low level tasks, meaning tasks with low cognitive demand. To keep the idea of “cognitive demand” central to our discussions throughout, tasks will be referred to as high cognitive demand tasks and low cognitive demand tasks.

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students are asked to solve several problems using the fact $3 \times 5 \times 7 = 105$. This problem can be classified a high cognitive demand task, and, more specifically, it is an example of one kind of task that involves developing understanding of procedures using connections. High cognitive demand tasks may:

- Focus students’ attention on the use of procedures for the purpose of developing deeper understanding of mathematical concepts and ideas.

- Suggest explicit and/or implicit pathways to follow that involve the use of broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms.

- Often be represented in multiple ways, including the use of manipulative materials, diagrams, and symbols. Making connections among the representations helps students develop meaning.

- Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students are engaged in conceptual ideas that underlie the procedure and develop understanding.

We might characterize this first task as one in which students’ number sense and fluency with fact knowledge are developed. In addition, for some students, depending on their experiences with mathematics, problems $b$ and $d - g$ may or may not make sense, that is, students may not have developed relational understanding regarding the use of the equal sign and, consequently, may provide incorrect solutions and/or be unable to solve the problems posed. In such a case, the original task now becomes a diagnostic tool, indicating the need to focus on how to interpret the equal sign as meaning one quantity “is the same as” another quantity.
The second task using knowledge of multiplication facts is a more traditional timed test of fact knowledge and is categorized as a low cognitive demand task, in this case, involving memorization. Low cognitive demand tasks:

- Involve reproducing previously learned facts, rules, formulas or definitions or committing these to memory.
- Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.
- Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated.
- Have no connection to the concepts or meanings that underlie the facts, rules, formulas, or definitions learned.

In the case of timed tests of multiplication facts, perhaps instruction concerning multiplication—what multiplication facts mean and how they are constructed, preceded this particular task or, as may often be the case, it could mean that students are being asked to memorize information about which they have had little opportunity to make sense. In either case, this particular task is a procedure without connections. In the former context, there may be a “place” for procedures without connections but careful thought needs to be given to where that place may be.

In Figure 5 are two different tasks that address learning that involves (among other things) knowledge of the area and perimeter formulas for a rectangle. Try completing each task as directed. What mathematics is involved? What might be the purpose for each task when used with 3rd or 4th grade students? What methods do you think students might employ for each task? What misconceptions might surface? What errors? If you categorize each task as either
high cognitive demand or low cognitive demand, where would each task fall? Why?

The first task involving the area and perimeter formulas may be classified as a high cognitive demand task that involves doing mathematics. Such tasks may:

- Require complex, non-algorithmic thinking.
- Require students to explore and understand the nature of mathematical concepts, processes, or relationships.
- Demand students do some type of self-monitoring or self-regulation of their own cognitive processes (i.e., metacognition).
- Require students to access relevant knowledge and experiences and make appropriate uses of them in working through the task (i.e., activating prior knowledge).
- Require students to analyze task constraints that may limit possible solution strategies or solutions.
- Require considerable cognitive effort and may cause some level of anxiety for the students as they are working through the problem.

As with any task, the knowledge brought by the student solving the problem must be considered in deciding a task’s classification. The Rabbit Pen task is an applied problem in which students are using and extending knowledge they have already developed. The second task—Beverly’s Carpet problem, appropriately follows instruction in which the area theorem has been developed using a variety of tasks, many of which may involve procedures with connections.
Figure 5: High cognitive demand? Low cognitive demand? Which is which?

1. Ms. William’s class is raising rabbits for their spring science fair. They have 24 feet of fencing to use to build a rectangular rabbit pen for the rabbits.
   a. If Ms. William’s students made a pen to give the rabbits as much room as possible, how long would each of the sides of the pens be?
   b. How long would each of the sides of the pen be if they only had 16 feet of fencing?
   c. How would you go about determining the pen with the most room for any amount of fencing? Organize your work so that someone else who reads it will understand it.

2. Beverly wants to re-carpet her bedroom. The room is 15 feet long and 10 feet wide. How many square feet of carpeting does she need to purchase?

The second problem may be classified as a low cognitive demand task that focuses on using procedures without connections. Such tasks:

- Are algorithmic. The use of a procedure either is specifically called for or is evident from prior instruction and/or experience.
- Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it.
  a. Are not connected to the concepts or meaning that underlie the procedure being used.
  b. Are focused on producing correct answers.
  c. Require no explanation or explanations focus solely on describing the procedure that was used.

The use of such a task may be appropriate within a sequence of instruction in which students completing the task have a well-developed understanding of the context and concepts related to the area and perimeter formulas and now are demonstrating their ability to apply the formulas within a limited problem context. When tasks like this one are used as a way to teach the area formula through direct instruction and then to practice a procedure with limited
or no development of conceptual understanding, the task continues to be classified as procedures without connections.

In addition to providing a taxonomy for classifying mathematical tasks, Stein and her colleagues (1996, 2000) also provide a framework for tracking the cognitive demands of mathematical tasks as they unfold during instruction and for exploring the connection between instruction and student learning. The Mathematical Tasks Framework (MTF) distinguishes three phases through which tasks pass as they unfold during a lesson (Stein, Grover, & Henningsen, 1996):

1. As they appear in curricular or instructional materials (i.e., on the printed pages of textbooks, ancillary materials, or as created by teachers);

2. As they are set up or announced or launched by the teacher;

3. As they are actually implemented by students and the teacher in the classroom, i.e., the way in which students actually go about working on the task,

And we have added a fourth phase to those noted above that is particularly relevant when high cognitive demand tasks are used:

4. As the mathematics to be learned is summarized in a group discussion by students and teachers in the classroom – in other words, how the mathematics to be learned is synthesized.
Figure 6: Modified Mathematical Task Framework

All of these, but especially the implementation and summary phases, are viewed as important influences on what students actually learn. Recent studies have shown that in classrooms where teachers frequently set up mathematical tasks that engage students in high cognitive demands, students learn more than their counterparts do in classrooms where tasks with low cognitive demands are used more frequently. (Henningsen, 2000, p. 244)

Additionally, in a recent review of research about the effects of classroom mathematics teaching on students’ learning, Hiebert and Grouws (2007) defined two features that emerged as promoting deep conceptual understanding: 1) teachers and students attend explicitly to concepts; and 2) students struggle with important mathematics. They refer to attending to concepts as, “treating mathematical connections is an explicit and public way (p. 383).” The public discussion that occurs at the summary phase of a mathematical task should address:

- The mathematical meaning of the procedures used in solving the task;
- How students’ solution methods are similar and different from each other;
- How the current task is related to other mathematical tasks the students have considered;
- How the mathematical ideas involved in the tasks are related; and
• How the current task relates to the mathematical concepts they are currently studying.

When Hiebert and Grouws (2007) refer to struggling with important mathematics, they do not suggest that teachers allow students to become frustrated by challenges that are beyond their current ability to cope. Rather, they join Polya (1973/1945) in noting that part of a successful mathematical experience for a student exists in letting them use their own curiosity to explore an interesting problem that is within their reach. During the summary phase of task analysis, the discussion of the problem and its solution among students and the teacher allows students to reflect on their own problem solving activities and compare them to the ideas and activities of their peers. It is during this period of self-reflection that students build mathematical ideas.

The research conducted by Stein and her colleagues suggests that the cognitive demands of a task can change between any two phases of the framework. For example, the task that appears in curricular or instructional materials is not always identical to the task that is set up by the teacher in the classroom, and this task, in turn, is not always identical to the task that the students actually do. Between the set-up and implementation phases, tasks can transform for a variety of reasons, many of which have to do with teachers’ beliefs concerning their students’ abilities to solve a task. For example, a teacher, in launching the third place value task (Figure 3), asks Question 4. “Can you think of another way to show 342 using these materials? How did you think about this?” and proceeds to show the children an example and explains why the new representation still equals 342. The teacher has “over-launched” the problem and removed much of the challenge. In doing so, this has changed the problem by reducing the cognitive demand. It is interesting to note that tasks that are classified as high cognitive demand may be implemented in ways that reduce the cognitive demand. However,
tasks that are classified as low level of cognitive demand initially are nearly always implemented as intended (Stein, Grover, & Henningsen, 1996).

It is useful to explore ways to create problems that support high cognitive demands. The following process can be helpful in developing high cognitive demand tasks for your students. Your tasks need to:

1. Clearly illustrate the mathematical concept you are trying to convey.
2. Be written unambiguously.
3. Describe situations that will be familiar and accessible to the students.
4. Be complex enough that they can be represented in more than one way.
5. Build on students’ prior knowledge.
6. Provide students with an attainable challenge.

In addition to relying on experience to help you develop tasks for your students, you may also access sources of problems and student work such as the National Assessment of Education Progress (NAEP) via books (Kloosterman & Lester, 2007) or websites such as, 


Remember that we are interested in students being able to develop robust notions and attitudes about problem solving. For such a problem solving mindset to develop in students, they need to be provided with opportunities to solve good problems and to be offered opportunities to develop their own problems. Students should not be denied the pleasure of posing clever tasks to one another!

References


Outhred, Lynne, & Sara Sardelich (October 2005). A problem is something you don’t want to have”: Problem solving by kindergarteners. *Teaching Children Mathematics* (146-154).


